Basics of Homotopy Groups with Coefficients

Riley Moriss

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Definition

Following [Hat02] and [Nei10]. For an abelian group A a Moore space $M(A,k)$ is a space that has *homology* groups A in degree k and *reduce homology groups* zero everywhere else [Hat02, Example 2.40]. It turns out that $M(\mathbb{Z}_p, k)$ also has *cohomology* $\mathbb{Z}_p$ in degree $k+1$, such a space is called a Peterson space $P(\mathbb{Z}_p, k+1)$. For $n > 1$ by [Hat02, Example 4.34] there is a unique *simply connected* Moore space up to homotopy for any given abelian group, for this reason it may be included in the definition that the Moore space be simply connected. In general $M(A,k) = \Sigma M(A, k-1)$, which can be seen from the suspension isomorphism on homology. We may denote the Moore space valued homotopy groups

$$\pi_k(X; A) \; := \; [M(A, k-1), X]_*$$

These are alternatively called homotopy groups with coefficients. When $A = \mathbb{Z}_p$ we see that alternitively $\pi_k(X;\mathbb{Z}_p) = [P(\mathbb{Z}_p, k), X]$, this dualises the well known fact that $H^i(X;\mathbb{Z}_p) = [X, K(\mathbb{Z}_p, i)]$; cohomology is mapping into a homotopically concentrated object and homotopy is mapping out of a cohomologically concentrated object. Hatcher gives a construction for a general Moore space however we will only really care about the cyclic groups.

Example. *It is well known that S^n is a $M(\mathbb{Z}, n)$, however S^n is not simply connected. One can see that uniqueness fails through the standard homology sphere examples.*

Example. *Let $A = \mathbb{Z}_p$. Then the Moore space $M(A,n)$ is S^n with a D^{n+1} disc glued to it along the discs boundary by a degree p map.*

Remark. Even though for an arbitrary abelian group we have a Moore space, a space whose homology is that group in some degree, we only have this in cohomology for $\mathbb{Z}_p$. As Hatcher points out there can be no such group for cohomology and say the group $\mathbb{Q}$.

Remark. The reason we use Moore space is that it allows us to extend the definition of homotopy groups with coefficients to any abelian group. On the groups for which Peterson spaces exist then they agree anyway (with a shift).

Sequences

There is a universal coefficients for a general abelian A group [Hat02, Prop 4H.2], for $n > 1$

$$0 \to \mathrm{Ext}(A, \pi_{n+1}(X)) \to \pi_{n+1}(X; A) \to \mathrm{Hom}(A, \pi_n(X)) \to 0$$

Warning, there is not a tor version of this universal coefficients (more below).

For $\mathbb{Z}_p$ we futher have the Puppe sequence for the cofibration [Hat02, Thm 4.58]

$$S^n \hookrightarrow S^n \cup_p D^{n+1} = M(\mathbb{Z}_p, n)$$

That sequence looks like

$$S^n \to M(\mathbb{Z}_p, n) \xrightarrow{\text{collapse}} M(\mathbb{Z}_p, n)/S^n = S^{n+1} \to \Sigma S^n \to \Sigma M(\mathbb{Z}_p, n) \to \Sigma S^{n+1} \to \Sigma^2 S^n \to \cdots$$

Note that the "boundary" map $M(\mathbb{Z}_p, n)/S^n = S^{n+1} \to \Sigma S^n$ is the degree p map. We can use our standard identities to simplify

$$S^n \to M(\mathbb{Z}_p, n) \to S^{n+1} \xrightarrow{p} S^{n+1} \to M(\mathbb{Z}_p, n+1) \to S^{n+2} \to S^{n+2} \to \cdots$$

We can apply $[-, X]$ to this sequence, this will reverse the arrows and give an exact sequence

$$\pi_n(X) \leftarrow \pi_{n+1}(X; \mathbb{Z}_p) \leftarrow \pi_{n+1}(X) \xleftarrow{\times p} \pi_{n+1}(X) \leftarrow \pi_{n+1}(\mathbb{Z}_p, X) \leftarrow \cdots$$

which by looking at the $n = 1$ case can be extended to the π_1 exactly to the left.

Remark. The reason that there is no tor universal coefficients is that the proof uses a fibration and the maps will go the wrong way, that is homotopy groups with coefficients mirror cohomology. In cohomology the tor universal coefficients come from the homology universal coefficients, which crucially uses that it is stable, that is LES for SES of chain complexes. maybe there is a tor for cohomotopy with coefficients that I can use? Anyway it seems unlikely to me now, because maps are going the wrong way and the dualisation is not clean without homology theory properties.

Lemmas

Here are a few extra sanity checks or facts that we will make use of elsewhere.

Lemma (Cor 6.6, [Nei10])**.** *$\pi_n(X; A)$ is a functor for $n \geq 4$ in the spaces and in finitely generated abelian groups with no two torsion (A). The functor lands in groups.*

Lemma.

$$\pi_i(\Omega X; \mathbb{Z}_p) = \pi_{i+1}(X; \mathbb{Z}_p)$$

Proof.

$$\pi_i(\Omega X; \mathbb{Z}_p) = [M(A, i-1), \Omega X] = [\Sigma M(A, i-1), X] = [M(A, i), X] = \pi_{i+1}(X; A)$$

References

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge ; New York, 2002.

[Nei10] Joseph A. Neisendorfer. Homotopy groups with coefficients. *Journal of Fixed Point Theory and Applications*, 8(2):247–338, December 2010.